Displacement current — Displacement current is the current which is setup in a dielectric medium ($\sigma=0$) due to variation of induced displacement charge produced by the changing electric field applied across of the dielectric.

Characteristics of displacement current :—

a) Displacement current is a current only in the sense that it produces a magnetic field. It has none of other properties of current.

b) Displacement current serves the purpose to make the total current continuous across the discontinuity in a conduction current.

c) The magnitude of displacement current density is the rate of change of electric displacement vector ie $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$.

d) In a good conductor $\vec{J}_d$ is negligible compared to $\vec{J}$.

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1. Starting with the diff. forms of Faraday's law in electromagnetic induction and Ampere's modified circuital law obtain the diff. forms of Gauss's law of magnetostatics and electrostatics respectively. V.U - 1994

i) Diff. form of Faraday's law in ~~electrostatics~~ magnetic induction,
$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

Taking divergence on both sides,
$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\cdot\frac{\partial \vec{B}}{\partial t}.$$
$$\Rightarrow \vec{\nabla}\cdot\frac{\partial \vec{B}}{\partial t} = 0$$ since div. curl of any vector is zero.

Since differentiation w.r.to. space and time are interchangable, $\therefore \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) = 0$.

Thus either $\vec{\nabla}\cdot\vec{B} = $ const or 0.

As isolated mag. poles do not exist in nature, thus $\vec{\nabla}\cdot\vec{B} \neq$ const

$\therefore \boxed{\vec{\nabla}\cdot\vec{B} = 0}$ Gauss's law in mag.

ii) Modified Ampere's circuital law,

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Taking divergence on both sides,

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\frac{\partial \vec{D}}{\partial t}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\frac{\partial \vec{D}}{\partial t} = 0$$ since div. curl of any vector is zero.

$$\Rightarrow \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D}) = -\vec{\nabla}\cdot\vec{J} = \frac{\partial \rho}{\partial t}$$ From eqn. of continuity $\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$

$$\therefore \boxed{\vec{\nabla}\cdot\vec{D} = \rho}$$

Gauss's law in electrostatics.

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Starting from Maxwell's equation for a homogeneous isotropic dielectric medium of conductivity σ derive an expression for the growth or decay of electric charge density and hence define relaxation time.

V.U - 1989.

Maxwell's equation, $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$.

Taking divergence on both sides,

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\cdot\vec{J} + \vec{\nabla}\cdot\left(\frac{\partial \vec{D}}{\partial t}\right)$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J} + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D}) = 0 \quad \text{——— (1)}$$ since div. curl of any vector is zero.

From Ohm's law $\vec{J} = \sigma\vec{E}$.

From Maxwell's 1st eqn. $\vec{\nabla}\cdot\vec{D} = \rho$.

$$\therefore \vec{\nabla}\cdot\sigma\vec{E} + \frac{\partial \rho}{\partial t} = 0.$$

$$\Rightarrow \sigma\,\rho/\epsilon + \frac{\partial \rho}{\partial t} = 0.$$

$\vec{\nabla}\cdot\vec{D} = \rho$

$\Rightarrow \vec{\nabla}\cdot\epsilon\vec{E} = \rho$

$\Rightarrow \vec{\nabla}\cdot\vec{E} = \rho/\epsilon$

$$\Rightarrow \frac{\partial \rho}{\partial t} = -\frac{\sigma}{\epsilon}\rho$$

$$\Rightarrow \int_{\rho_0}^{\rho} \frac{\partial \rho}{\rho} = -\frac{\sigma}{\epsilon}\int_0^t dt \cdot \text{ Integretting,}$$

$$\Rightarrow \ln \rho/\rho_0 = -\frac{\sigma t}{\epsilon}.$$

$$\Rightarrow \boxed{\rho = \rho_0 e^{-\sigma t/\epsilon}}.$$

Thus charge density decays exponentially with time.

Relaxation time –

At $t = \tau$, $\rho = \rho_0/e$

Then $\frac{\rho_0}{e} = \rho_0 e^{-\frac{\sigma}{\epsilon}\tau}$

$$\Rightarrow \frac{\sigma}{\epsilon}\tau = 1$$

$$\Rightarrow \boxed{\tau = \frac{\epsilon}{\sigma}}.$$

(3) Show that equation of continuity $\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0$ is contained in Maxwell's equation. (V.U - 1995, 2001)

From Maxwell's 4th equation,

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}.$$

Taking divergence on both sides,

$$\vec{\nabla}.(\vec{\nabla}\times\vec{H}) = \vec{\nabla}.\vec{J} + \vec{\nabla}.\frac{\partial \vec{D}}{\partial t}.$$

$$\Rightarrow \vec{\nabla}.\vec{J} + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{D}) = 0$$

since div. curl of any vector is zero. space and time operations are interchangable.

$$\Rightarrow \boxed{\vec{\nabla}.\vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

Since $\vec{\nabla}.\vec{D} = \rho$.

we get equation of continuity.

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① State and Establish Poynting Theorem —

Statement — It states that time rate of change of electromagnetic energy within a certain volume plus the time rate of ~~change~~ of energy flowing out through the boundary surface is equal to the power transffered into the e.m. field.

ie $$\frac{\partial}{\partial t}\iiint u\,dv + \oint\!\!\oint_S \vec{P}\cdot\hat{n}\,ds = -\iiint \vec{J}\cdot\vec{E}\,dv.$$

or $$\oint\!\!\oint (\vec{E}\times\vec{H})\cdot\hat{n}\,ds + \frac{\partial}{\partial t}\iiint \frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B})\,dv = -\iiint \vec{J}\cdot\vec{E}\,dv.$$

Proof —

Electrostatic energy density $u_e = \frac{1}{2}\vec{D}\cdot\vec{E}$.

Magnetostatics ~~Electro-Magnetic~~ energy density $u_m = \frac{1}{2}\vec{B}\cdot\vec{H}$.

Thus electromagnetic energy density,

$$u = u_e + u_m = \frac{1}{2}(\vec{D}\cdot\vec{E}+\vec{B}\cdot\vec{H}).$$

We have Maxwell's equations —

$$\vec{\nabla}\cdot\vec{D} = \rho \quad \text{—— ①}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{—— ②}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{—— ③}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \quad \text{—— ④}$$

Again $\vec{B} = \mu\vec{H}$ and $\vec{D} = \epsilon\vec{E}$.

Taking scalar product of equation ③ with $\vec{H}$ and equation ④ with $\vec{E}$, we get ——

$$\vec{H}\cdot(\vec{\nabla}\times\vec{E}) = -\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} \quad \text{——} (5)$$
$$\vec{E}\cdot(\vec{\nabla}\times\vec{H}) = \vec{E}\cdot\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right) \quad \text{——} (6)$$

*) (i) Subtracting equation (6) from equation (5), we get –

$$\vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H}) = -\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}+\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right) - \vec{J}\cdot\vec{E}$$
$$\Rightarrow \vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\left(\vec{H}\cdot\mu\frac{\partial\vec{H}}{\partial t}+\vec{E}\cdot\epsilon\frac{\partial\vec{E}}{\partial t}\right) - \vec{J}\cdot\vec{E}$$
$$\Rightarrow \vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\frac{1}{2}\mu\frac{\partial}{\partial t}(\vec{H}\cdot\vec{H}) - \frac{1}{2}\epsilon\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E}) - \vec{J}\cdot\vec{E}$$
$$\Rightarrow \vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\frac{1}{2}\frac{\partial}{\partial t}(\vec{B}\cdot\vec{H}) - \frac{1}{2}\frac{\partial}{\partial t}(\vec{D}\cdot\vec{E}) - \vec{J}\cdot\vec{E}$$
$$\Rightarrow \boxed{\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \frac{1}{2}\frac{\partial}{\partial t}\left[\vec{B}\cdot\vec{H}+\vec{E}\cdot\vec{D}\right] = -\vec{J}\cdot\vec{E}}$$

V.U – '99, 2004

*) (ii) Subtracting equation (6) from (5), we get –

$$\vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{H}) = -\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}+\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right) - \vec{J}\cdot\vec{E}$$

or $$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}+\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right) = -\vec{J}\cdot\vec{E} \quad \text{——} (7)$$

Again
$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{2}(\vec{D}\cdot\vec{E}+\vec{B}\cdot\vec{H})\right]$$
$$= \frac{1}{2}\left[\frac{\partial\vec{D}}{\partial t}\cdot\vec{E} + \vec{D}\cdot\frac{\partial\vec{E}}{\partial t} + \frac{\partial\vec{B}}{\partial t}\cdot\vec{H} + \vec{B}\cdot\frac{\partial\vec{H}}{\partial t}\right]$$
$$= \cancel{\frac{1}{2}}\left[\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right] \quad \text{as } \vec{D}=\epsilon\vec{E},\ \vec{B}=\mu\vec{H}$$
$$\text{——} (8)$$

Combining eqn. (7) and (8),

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \frac{\partial u}{\partial t} = -\vec{J}\cdot\vec{E}$$

Multiplying both sides by dv and integrating over the volume,

$$\iiint \vec{\nabla}\cdot(\vec{E}\times\vec{H})\,dv + \iiint\frac{\partial u}{\partial t}\,dv = -\iiint\vec{J}\cdot\vec{E}\,dv$$

or, $$\boxed{\oiint(\vec{E}\times\vec{H})\cdot\hat{n}\,ds + \frac{\partial}{\partial t}\iiint u\,dv = -\iiint\vec{J}\cdot\vec{E}\,dv}$$ { According to Gauss's theorem.

Proved

Differential form of Poynting Theorem

We have Poynting theorem –

$$\text{div.}\left[\vec{E}\times\vec{H}\right] = -\frac{1}{2}\frac{\partial}{\partial t}\left[\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H}\right] - \vec{J}\cdot\vec{E}.$$

Taking volume integral,

$$\iiint_V \vec{\nabla}\cdot\left[\vec{E}\times\vec{H}\right]dV = -\frac{\partial}{\partial t}\iiint_V \frac{1}{2}\left[\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H}\right]dV - \iiint_V \vec{J}\cdot\vec{E}\,dV$$

$$\Rightarrow \iiint \vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right)dV = -\iiint \frac{\partial u}{\partial t}dV - \iiint \vec{J}\cdot\vec{E}\,dV$$

$$\Rightarrow \iiint \left[\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) + \frac{\partial u}{\partial t} + \vec{J}\cdot\vec{E}\right]dV = 0.$$

Since volume taken is arbitary, $dV \neq 0$.

$$\therefore \frac{\partial u}{\partial t} + \vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) + \vec{J}\cdot\vec{E} = 0.$$

Equation of continuity (V.U - '03).

For a medium having conductivity zero ie $\sigma = 0$.

$$\therefore \vec{J} = \sigma\vec{E} = 0.$$

Then $\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\left(\vec{E}\times\vec{H}\right) = 0$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{P} = 0}$$

Where $\vec{P} = \vec{E}\times\vec{H}$, is Poynting vector.

We get equation of continuity.

Poynting vector

$\vec{P}$ or $\vec{S} = \vec{E}\times\vec{H}$ is known as Poynting vector. Which is interpreted as power flux ie amount of energy ~~of the~~ crossing unit area placed perpendicular to the vector, per unit time.

Average value of Poynting vector

The average value of Poynting vector represents the intensity of the e.m. wave.

i.e Intensity $I = \langle \vec{S} \rangle = \langle \vec{E} \times \vec{H} \rangle = \frac{1}{2} E_0 \cdot H_0$.

~~Plane electromagnetic waves in free space~~

~~Maxwell's equation~~

Dimension of Poynting vector

$F = qE$

$H = i/l = \frac{q}{tl}$

Poynting vector $\vec{S} = \vec{E} \times \vec{H}$.

$$\therefore [\vec{S}] = [E][H] = \frac{[F]}{[q]} \times \frac{[q]}{[l][t]}$$

$$= \frac{MLT^{-2}}{LT} = \underline{[MT^{-3}]}$$

Unit $\Rightarrow$ Watt/m²

Plane electromagnetic waves in free space

Maxwell's equations are —

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

and

$$\vec{B} = \mu \vec{H}$$

$$\vec{D} = \epsilon \vec{E}$$

$$\vec{J} = \sigma \vec{E}$$

For free space, $\rho = 0$, $\sigma = 0$, $\mu = \mu_0$, $\epsilon = \epsilon_0$.

Thus Maxwell's equations in free space,

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \text{——— (1)}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \quad \text{——— (2)}$$

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \quad \text{——— (3)}$$

$$\vec{\nabla} \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \text{——— (4)}$$

Taking curl of equation (3), we get,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu_0 \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H}).$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu_0 \frac{\partial}{\partial t}\left(\epsilon_0 \frac{\partial\vec{E}}{\partial t}\right) \quad \text{from eqn. (4)}.$$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu_0\epsilon_0 \frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\mu_0\epsilon_0 \frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow -\nabla^2\vec{E} = -\mu_0\epsilon_0 \frac{\partial^2\vec{E}}{\partial t^2} \quad \text{from eqn. (1) } \vec{\nabla}\cdot\vec{E}=0.$$

$$\Rightarrow \boxed{\nabla^2\vec{E} - \mu_0\epsilon_0 \frac{\partial^2 E}{\partial t^2} = 0} \quad \text{——— (5)}$$

Similarly, taking curl of equation (4), we get —

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}).$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{H}) - \nabla^2\vec{H} = -\epsilon_0\mu_0 \frac{\partial^2\vec{H}}{\partial t^2} \quad \text{as } \vec{\nabla}\times\vec{E} = -\mu_0\frac{\partial\vec{H}}{\partial t}$$

$$\Rightarrow -\nabla^2\vec{H} = -\epsilon_0\mu_0 \frac{\partial^2\vec{H}}{\partial t^2} \quad \text{as } \vec{\nabla}\cdot\vec{H} = 0.$$

$$\Rightarrow \boxed{\nabla^2\vec{H} - \epsilon_0\mu_0 \frac{\partial^2\vec{H}}{\partial t^2} = 0} \quad \text{——— (6)}$$

Equation (5) and (6) represents wave equations governing e.m. field E and H in free space.

Similarly scalar wave equation,

$$\nabla^2 u - \mu_0\epsilon_0 \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{——— (7)}$$

where u stands for one of the components of E and H.

Comparing with the eqn. $\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$.

$$v = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 3\times10^8 \text{ m/s} = c, \text{ speed of light}$$